WEAK ASYMPTOTIC PROPERTIES OF PARTITIONS

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1. Introduction. Let Λ be a denumerable set of distinct positive numbers without finite limit point. Then the elements of the set can be arranged in a sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, where λ_k tends to infinity with k. Let $0 = \nu_0 < \nu_1 < \nu_2 < \cdots$ be the elements of the additive semi-group generated by Λ. Let $p(\nu_i)$ denote the number of ways of expressing ν_i in the form $m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + \cdots$, where the m_j are non-negative integers. The generating function $f(s)$ of $p(\nu_m)$ is given by

$$f(s) = \prod_{k=1}^{\infty} (1 - e^{-s\lambda_k})^{-1} = \sum_{m=0}^{\infty} p(\nu_m) e^{-s\nu_m}.$$

We are concerned only with sets Λ which are such that both the product and the sum converge for all $s > 0$.

More generally we can consider the weighted partition function $p(\nu_m)$ defined by the generating function

$$f(s) = \prod_{k=1}^{\infty} (1 - e^{-s\lambda_k})^{-\psi(k)} = \sum_{m=0}^{\infty} p(\nu_m) e^{-s\nu_m},$$

where $\psi(k)$ is a function on the positive integers into the non-negative reals such that the sum and product are both convergent for all $s > 0$. (Actually convergence of either the sum or product implies convergence of the other.) Here

$$p(\nu_i) = \sum_{m_1\lambda_1 + m_2\lambda_2 + \cdots = \nu_i} \prod_{k=1}^{\infty} \binom{\psi(k) + m_k - 1}{m_k}$$

where the summation is taken over non-negative integers m_k, and is finite. The kth factor in the product is equal to 1 for all k such that $\lambda_k > \nu_i$. When $\psi(k) = 1$ for all positive integers k, this reduces to the case discussed in the preceding paragraph.

Letting $n(u) = \sum_{\lambda_k \leq u} \psi(k)$, we have

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$$\begin{aligned}\log f(s) &= \sum_{k=1}^{\infty} \psi(k) \log (1 - e^{-s\lambda_k})^{-1} \\ &= \sum_{k=2}^{\infty} \{n(\lambda_k) - n(\lambda_{k-1})\} \log (1 - e^{-s\lambda_k})^{-1} + n(\lambda_1) \log (1 - e^{-s\lambda_1})^{-1} \\ &= \sum_{k=1}^{\infty} n(\lambda_k)\{\log (1 - e^{-s\lambda_k})^{-1} - \log (1 - e^{-s\lambda_{k+1}})^{-1}\} \\ &= \sum_{k=1}^{\infty} n(\lambda_k) s \int_{\lambda_k}^{\lambda_{k+1}} \frac{e^{-us}}{1 - e^{-us}} du \\ &= s \int_{\lambda_1}^{\infty} \frac{e^{-us}}{1 - e^{-us}} n(u) du,\end{aligned}$$

where the above computation is justified if $n(\lambda_{k-1}) \log (1 - e^{-s\lambda_k})^{-1}$ tends to zero as k tends to infinity. This is in fact the case since

$$\begin{aligned}0 \leqq n(\lambda_{k-1}) \log (1 - e^{-s\lambda_k})^{-1} &= \sum_{m=1}^{k-1} \psi(m) \log (1 - e^{-s\lambda_k})^{-1} \\ &\leqq \sum_{m=1}^{k_0} \psi(m) \log (1 - e^{-s\lambda_k})^{-1} + \sum_{m=k_0+1}^{\infty} \psi(m) \log (1 - e^{-s\lambda_m})^{-1},\end{aligned}$$

where k_0 can be chosen so that $\sum_{m=k_0+1}^{\infty} \psi(m) \log (1-e^{-s\lambda_m})^{-1} < \epsilon$ (in view of our assumption about ψ) and where $\sum_{m=1}^{k_0} \psi(m) \log (1-e^{-s\lambda_k})^{-1} \to 0$ as $k \to \infty$.

Now letting $P(u) = \sum_{\nu_i \leqq u} p(\nu_i)$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} p(\nu_m) e^{-s\nu_m} &= \sum_{m=0}^{\infty} \{P(\nu_m) - P(\nu_{m-1})\} e^{-s\nu_m} \\ &= \sum_{m=0}^{\infty} P(\nu_m)\{e^{-s\nu_m} - e^{-s\nu_{m+1}}\} = s \int_0^{\infty} P(u) e^{-su} du,\end{aligned}$$

where the computation is justified as in the preceding paragraph.

Thus we have the following basic relation between the functions $n(u)$ and $P(u)$:

$$\exp \left\{ s \int_0^{\infty} \frac{e^{-su}}{1 - e^{-su}} n(u) du \right\} = s \int_0^{\infty} P(u) e^{-su} du. \tag{1}$$

The object of this paper is to show that an asymptotic relation of the form $n(u) \sim u^{\alpha} L(u)$ as $u \to \infty$, where α is a positive constant and L is a slowly oscillating function in the sense of Karamata [6](2), is equivalent to an asymptotic relation of the form $\log P(u) \sim u^{\alpha/(\alpha+1)} L^*(u)$, where L^* is a slowly

(2) Numbers in brackets refer to the bibliography.

oscillating function, related to L by a certain implicit formula which can be solved for L^* in the cases usually encountered. The special case of this where $L(u)$ is a constant was given by Knopp [7] and Erdös [3]. When $L(u)$ is a power of log u the deduction in one direction (from $n(u)$ to $P(u)$) was made by Brigham [1] using results of Hardy and Ramanujan [5].

When we start from the assumption about $n(u)$ we get an asymptotic formula only for log $P(u)$ and not for $P(u)$ itself, so that the results are weak in this sense. However it should be noted that the assumption made on $n(u)$ is a very simple and natural one, and the resulting asymptotic law which we obtain for log $P(u)$ is *equivalent* to the original assumption made for $n(u)$.

The proof of the above equivalence uses only the obvious monotonicity of $n(u)$ and $P(u)$ and the basic relation (1). Thus, more generally, starting with equation (1), where $n(u)$ and $P(u)$ are functions on the non-negative real numbers such that

$$\int_0^R \frac{n(u)}{u}\,du, \qquad \int_0^R \frac{n(u)\log u}{u}\,du, \qquad \text{and} \qquad \int_0^R P(u)du$$

exist in the Lebesgue sense for every positive R, and α and $L^*(u)$ are as before, we obtain the following:

(i) If $n(u) \sim u^\alpha L(u)$ as $u \to \infty$ and $P(u)$ is nondecreasing then log $P(u) \sim u^{\alpha/(\alpha+1)} L^*(u)$ as $u \to \infty$.

(ii) If log $P(u) \sim u^{\alpha/(\alpha+1)} L^*(u)$ as $u \to \infty$ and $n(u)$ is nondecreasing, then $n(u) \sim u^\alpha L(u)$ as $u \to \infty$.

Thus most of our results will be stated for functions $n(u)$ and $P(u)$ satisfying the foregoing conditions (for all positive s), and we shall specialize to the case of partitions only at the end of the paper.

To prove (i) we go from the assumption on $n(u)$ to a resulting property of the generating function $f(s)$ by an Abelian argument, and then from this property of $f(s)$ to the assertion about $P(u)$ by a Tauberian argument. The latter argument follows the method developed by Hardy and Ramanujan [5], for the special case in which the slowly oscillating function is a power of the logarithm.

To prove (ii) we go from the assumption on $P(u)$ to a resulting property of $f(s)$ by an Abelian argument and then from this property of $f(s)$ to the assertion about $n(u)$ by a Tauberian argument. The latter step is accomplished in two stages, in order to make use of a known Tauberian Theorem [4, Theorem 108].

The material in this paper is essentially the same as that contained in the author's doctoral dissertation at the University of Illinois under the direction of P. T. Bateman, to whom the author is deeply grateful.

2. **Definitions and properties of slowly oscillating functions.** Let the function $L(x)$ be defined and positive valued for all sufficiently large x, and let it

be continuous and such that $L(cx)/L(x)\to 1$ as $x\to\infty$ for every fixed positive c. Then $L(x)$ is called a *slowly oscillating function.*

We shall make use of the following properties of slowly oscillating functions:

(i) $L(x)=K(1+r(x))\exp\int_{x_0}^{x}(\delta(t)/t)dt$ where K is a positive constant, while $r(x)$ and $\delta(x)$ are continuous functions, which tend to zero as x tends to infinity. See [6; 8]. In the sequel in order to assure the differentiability of $L(x)$, we will occasionally assume that it is exactly (and not merely asymptotically) of the form $K\exp\int_{x_0}^{x}(\delta(t)/t)dt$. Whenever this assumption is made it will involve no loss of generality. A slowly oscillating function of this special form will be called a *normalized* slowly oscillating function and will be denoted by J.

(ii) The convergence $L(cx)/L(x)\to 1$ as $x\to\infty$ is uniform in c, for c in any finite interval $[a, b]$ with $0<a<b$. See [2; 6; 8].

It is easy to see that the following properties are consequences of the above.

(iii) $x^{\epsilon}L(x)\to\infty$ and $x^{-\epsilon}L(x)\to 0$ as $x\to\infty$ for every fixed positive ϵ.

(iv) $\log L(x)/\log x\to 0$ as $x\to\infty$. A proof of this is also to be found in Pólya-Szegö [10, p. 68]. There $L(x)$ is assumed to be nondecreasing, but this is not necessary for the proof.

We will require the following lemma.

LEMMA 1. *If α is a positive constant and $L(x)$ is a slowly oscillating function defined for $x\geqq x_0$, then*

$$\int_{x_0}^{x} t^{\alpha-1}L(t)dt \sim x^{\alpha}L(x)/\alpha \quad as \quad x\to\infty.$$

Proof. By (iii) both sides of the proposed asymptotic relation tend to infinity with x. Hence it suffices to prove this relation with L replaced by J, where J is the normalized slowly oscillating function associated with L as in (i). But on integration by parts we have

$$\alpha\int_{x_0}^{x} t^{\alpha-1}J(t)dt = x^{\alpha}J(x) - x_0^{\alpha}J(x_0) - \int_{x_0}^{x}\delta(t)t^{\alpha-1}J(t)dt$$

and so

$$\left|\alpha\int_{x_0}^{x} t^{\alpha-1}J(t)dt - x^{\alpha}J(x)\right| \tag{2}$$
$$\leqq x_0^{\alpha}J(x_0) + \int_{x_0}^{x_1}|\delta(t)|t^{\alpha-1}J(t)dt + \left\{\max_{t\geqq x_1}|\delta(t)|\right\}\int_{x_1}^{x} t^{\alpha-1}J(t)dt$$

where $x_0<x_1$. Now given an arbitrary positive number ϵ, we can choose x_1, so that $|\delta(t)|\leqq\epsilon$ for $t\geqq x_1$. Thus the right hand side of (2) is less than

$$2\epsilon\int_{x_0}^{x} t^{\alpha-1}J(t)dt$$

for sufficiently large x. Since ϵ is arbitrary, the lemma is established.

3. **The connection between $n(u)$ and $\log f(s)$:**

THEOREM 1. *If $\int_0^R n(u)u^{-1}du$ exists in the Lebesgue sense for every $R>0$ and $f(s)=\exp\left\{s\int_0^\infty (e^{-su}/(1-e^{-su}))n(u)du\right\}$, for all positive s, then the relation $n(u)\sim u^\alpha L(u)$ as $u\rightarrow\infty$ implies the relation*

$$\log f(s) \sim \Gamma(\alpha+1)\zeta(\alpha+1)(1/s)^\alpha L(1/s) \quad as \quad s\rightarrow 0,$$

where α is a positive constant and L is a slowly oscillating function for $u\geqq u_0$. (Here $\zeta(\alpha+1)=\sum_{m=1}^\infty 1/m^{\alpha+1}$ and $\Gamma(\alpha+1)=\int_0^\infty e^{-t}t^\alpha dt$.)

Proof. We first remark that if $G(u)=\int_u^\infty (v^\alpha/(e^v-1))dv$ then there is a constant K_1, such that $G(u)<K_1u^\alpha e^{-u}$ for $u\geqq 1$. For

$$G(u) < 2\int_u^{2u} v^\alpha e^{-v}dv + 2\int_{2u}^\infty e^{-v/2}(e^{-v/2}v^\alpha)dv$$

$$\leqq 2(2u)^\alpha\int_u^{2u} e^{-v}dv + 2e^{-u}\int_{2u}^\infty v^\alpha e^{-v/2}dv \leqq K_1u^\alpha e^{-u}.$$

Now let ϵ be an arbitrary positive number, less than min $(1, 1/3K_1)$. Since

$$\int_0^\infty \frac{u^\alpha}{e^u-1}du = \sum_{m=1}^\infty\int_0^\infty u^\alpha e^{-mu}du = \zeta(\alpha+1)\Gamma(\alpha+1)$$

we may choose a positive η so small that $5\eta^\alpha/\alpha<\epsilon$ and a positive γ so large that

$$\left|\zeta(\alpha+1)\Gamma(\alpha+1) - \int_\eta^\gamma \frac{u^\alpha}{e^u-1}du\right| < \epsilon. \tag{3}$$

Let us write

$$\log f(s) = \left(\int_0^\eta + \int_\eta^\gamma + \int_\gamma^\infty\right)\frac{n(u/s)}{e^u-1}du = I_1 + I_2 + I_3.$$

We now choose t_0 so large that $|n(t)|\leqq 2t^\alpha L(t)$ for $t\geqq t_0$. Then

$$|I_1| \leqq \left|s\int_0^{\eta/s}\frac{n(t)}{e^{st}-1}dt\right| \leqq \int_0^{\eta/s}\frac{|n(t)|}{t}dt$$

$$\leqq \int_0^{t_0}\frac{|n(t)|}{t}dt + 2\int_{t_0}^{\eta/s} t^{\alpha-1}L(t)dt = K_2 + 2\int_{t_0}^{\eta/s} t^{\alpha-1}L(t)dt.$$

Thus for sufficiently small s, we have by Lemma 1, the definition of a slowly oscillating function, and the choice of η

$$|I_1| \leqq K_2 + \frac{3}{\alpha}\left(\frac{\eta}{s}\right)^\alpha L\left(\frac{\eta}{s}\right) \leqq K_2 + \frac{4}{\alpha}\,\eta^\alpha\left(\frac{1}{s}\right)^\alpha L\left(\frac{1}{s}\right) < \epsilon\left(\frac{1}{s}\right)^\alpha L\left(\frac{1}{s}\right).$$

Let s be so small that for $u \geqq \eta$ we have

$$(1-\epsilon)\left(\frac{u}{s}\right)^\alpha L\left(\frac{u}{s}\right) \leqq n\left(\frac{u}{s}\right) \leqq (1+\epsilon)\left(\frac{u}{s}\right)^\alpha L\left(\frac{u}{s}\right).$$

Then

$$(1-\epsilon)L\left(\frac{1}{s}\right)\int_\eta^\gamma \frac{(u/s)^\alpha L(u/s)}{(e^u-1)L(1/s)}\,du$$
$$\leqq I_2 \leqq (1+\epsilon)L\left(\frac{1}{s}\right)\int_\eta^\gamma \frac{(u/s)^\alpha L(u/s)}{(e^u-1)L(1/s)}\,du.$$

Now, since $L(u/s)/L(1/s) \to 1$ as $s \to 0$, uniformly for $\eta \leqq u \leqq \gamma$, we have for s sufficiently small

$$(1-2\epsilon)\left(\frac{1}{s}\right)^\alpha L\left(\frac{1}{s}\right)\int_\eta^\gamma \frac{u^\alpha}{e^u-1}\,du$$
$$\leqq I_2 \leqq (1+2\epsilon)\left(\frac{1}{s}\right)^\alpha L\left(\frac{1}{s}\right)\int_\eta^\gamma \frac{u^\alpha}{e^u-1}\,du.$$

Using (3) this becomes

$$(1-2\epsilon)\left(\frac{1}{s}\right)^\alpha L\left(\frac{1}{s}\right)\left\{\zeta(\alpha+1)\Gamma(\alpha+1)-\epsilon\right\}$$
$$\leqq I_2 \leqq (1+2\epsilon)\left(\frac{1}{s}\right)^\alpha L\left(\frac{1}{s}\right)\left\{\zeta(\alpha+1)\Gamma(\alpha+1)+\epsilon\right\}.$$

By property (i) of a slowly oscillating function we have for sufficiently small s,

$$I_3 \leqq 2\int_\gamma^\infty \frac{(u/s)^\alpha L(u/s)}{e^u-1}\,du \leqq 3\left(\frac{1}{s}\right)^\alpha \int_\gamma^\infty \frac{u^\alpha}{e^u-1}\,J\left(\frac{u}{s}\right)du,$$

where J is the normalized slowly oscillating function associated with L as in property (i) of a slowly oscillating function. Now

$$\int_\gamma^\infty \frac{u^\alpha}{e^u-1}\,J\left(\frac{u}{s}\right)du = -\,G(u)J\left(\frac{u}{s}\right)\Big|_\gamma^\infty + \int_\gamma^\infty G(u)\delta\left(\frac{u}{s}\right)u^{-1}J\left(\frac{u}{s}\right)du$$
$$\leqq G(\gamma)J\left(\frac{\gamma}{s}\right) + \left\{\max_{u\geqq\gamma}\left|\delta\left(\frac{u}{s}\right)\right|\right\}\int_\gamma^\infty G(u)J\left(\frac{u}{s}\right)u^{-1}du$$
$$\leqq 2\epsilon J\left(\frac{1}{s}\right) + \epsilon K_1\int_\gamma^\infty \frac{u^\alpha}{e^u-1}\,J\left(\frac{u}{s}\right)du$$

for s sufficiently small, since $G(u) < K_1 \; u^\alpha e^{-u} < K_1 u^\alpha/(e^u - 1)$ for $u \geqq 1$ and since $|G(\gamma)| \leqq \epsilon$ by (3). Hence, by the restriction $\epsilon < 1/3K_1$, we have for s sufficiently small

$$\int_\gamma^\infty \frac{u^\alpha}{e^u - 1} J\left(\frac{u}{s}\right) du \leqq \frac{2\epsilon}{1 - K_1\epsilon} J\left(\frac{1}{s}\right) < 3\epsilon J\left(\frac{1}{s}\right) < 4\epsilon L\left(\frac{1}{s}\right),$$

so that $I_3 \leqq 12\epsilon(1/s)^\alpha L(1/s)$.

Since ϵ is arbitrary, our result follows from the estimates for I_1, I_2 and I_3.

The Tauberian counterpart of Theorem 1 is:

THEOREM 2. *If $\int_0^R n(u)u^{-1}du$ and $\int_0^R n(u)u^{-1} \log u du$ exist in the Lebesgue sense, for every positive R, $f(s) = \exp \left\{ s\int_0^\infty (e^{-su}/(e^{-su} - 1))n(u)du \right\}$ for all positive s, and $n(u)$ is nondecreasing, then the relation*

$$\log f(s) \sim \zeta(\alpha + 1)\Gamma(\alpha + 1)(1/s)^\alpha L(1/s)$$

as $s \to 0$ implies the relation $n(u) \sim u^\alpha L(u)$ as $u \to \infty$, where α is a positive constant and L is a slowly oscillating function for $u \geqq u_0$.

Proof. We give the proof in two stages in order to make use of a known Tauberian Theorem. Define $N(u) = \sum_{m=1}^\infty (1/m)n(u/m)$. This exists, since

$$\sum_{m=2}^\infty \frac{1}{m} n\left(\frac{u}{m}\right) < \int_1^\infty \frac{1}{x} n\left(\frac{u}{x}\right) dx = \int_0^u \frac{n(t)}{t} dt, \tag{4}$$

the non-negativity of $n(u)$ being implied by the hypothesis. Clearly $N(u)$ is a non-negative, nondecreasing function of u. Now

$$\begin{aligned} \log f(s) &= s\int_0^\infty \frac{e^{-su}}{e^{-su} - 1} n(u)du \\ &= \sum_{m=1}^\infty s\int_0^\infty e^{-mus} n(u)du \\ &= \sum_{m=1}^\infty s\int_0^\infty e^{-su} \frac{1}{m} n\left(\frac{u}{m}\right) du \\ &= s\int_0^\infty e^{-su} N(u)du. \end{aligned}$$

To this last expression we apply [4, Theorem 108] and obtain $N(u) \sim \zeta(\alpha+1)u^\alpha L(u)$ as $u \to \infty$.

It remains to show that the relation $N(u) \sim \zeta(\alpha+1)u^\alpha L(u)$ as $u \to \infty$ implies the relation $n(u) \sim u^\alpha L(u)$ as $u \to \infty$, under the assumptions of the theorem.

From the definition of $N(u)$ and the properties of the Möbius function we obtain

$$n(u) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m} N\left(\frac{u}{m}\right).$$

This requires proving that

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mu(m)}{mk} n\left(\frac{u}{mk}\right)$$

is absolutely convergent. This is the case, since by (4) we have

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mk} n\left(\frac{u}{mk}\right) &\leqq \sum_{k=1}^{\infty} \left\{\frac{1}{k} n\left(\frac{u}{k}\right) + \frac{1}{k} \int_0^{u/k} \frac{n(t)}{t} dt\right\} \\ &= N(u) + \int_0^u \left(\sum_{1 \leqq k \leqq u/t} \frac{1}{k}\right) \frac{n(t)}{t} dt \\ &\leqq N(u) + \int_0^u \left(1 + \log \frac{u}{t}\right) \frac{n(t)}{t} dt, \end{aligned}$$

and these integrals exist by hypothesis.

Let ϵ be an arbitrary positive number less than $1/\zeta(\alpha+1)$. Choose $m_0 \geqq 2$ so large that $\left| \sum_{k=m_0+1}^{\infty} \mu(k)/k^{\alpha+1} \right| < \epsilon$ and $(1/m_0)^\alpha < \epsilon/3$. Choose u_0 so large that

$$(1 - \epsilon)\left(\frac{u}{k}\right)^\alpha L\left(\frac{u}{k}\right)\zeta(\alpha + 1) \leqq N\left(\frac{u}{k}\right) \leqq (1 + \epsilon)\left(\frac{u}{k}\right)^\alpha L\left(\frac{u}{k}\right)\zeta(\alpha + 1)$$

and

$$\frac{(1 - 2\epsilon)}{1 - \epsilon} L(u) \leqq L\left(\frac{u}{k}\right) \leqq \frac{(1 + 2\epsilon)}{1 + \epsilon} L(u) \text{ for } u \geqq u_0 \text{ and } 1 \leqq k \leqq m_0.$$

Then

$$\begin{aligned} &\left| \sum_{k=1}^{m_0} \frac{\mu(k)}{k} N\left(\frac{u}{k}\right) - u^\alpha L(u)\zeta(\alpha + 1) \sum_{k=1}^{m_0} \frac{\mu(k)}{k^{\alpha+1}} \right| \\ &\qquad \leqq 2\epsilon \sum_{k=1}^{m_0} \frac{|\mu(k)|}{k^{\alpha+1}} u^\alpha L(u)\zeta(\alpha + 1) < 2\epsilon\zeta^2(\alpha + 1)u^\alpha L(u). \end{aligned}$$

Since $\sum_{k=1}^{\infty} (u(k)/k^{\alpha+1}) = 1/\zeta(\alpha+1)$, our choice of m_0 guarantees that

$$\left| \sum_{k=1}^{m_0} \frac{\mu(k)}{k} N\left(\frac{u}{k}\right) - u^\alpha L(u) \right| < 3\epsilon\zeta^2(\alpha + 1)u^\alpha L(u).$$

Choose t_0 so large that $|N(t)| \leqq 2t^\alpha L(t)\zeta(\alpha+1)$ for $t \geqq t_0$. Then for sufficiently large u

$$\left|\sum_{k=m_0+1}^{\infty} \frac{\mu(k)}{k} N\left(\frac{u}{k}\right)\right| < \int_{m_0}^{\infty} \frac{N(u/x)}{x} dx = \int_0^{t_0} + \int_{t_0}^{u/m_0} N(t)t^{-1}dt$$

$$\leqq \int_0^{t_0} N(t)t^{-1}dt + 2\zeta(\alpha+1)\int_{t_0}^{u/m_0} t^{\alpha-1}L(t)dt \leqq \frac{3}{\alpha}\left(\frac{u}{m_0}\right)^{\alpha} L\left(\frac{u}{m_0}\right)\zeta(\alpha+1)$$

$$< \epsilon u^{\alpha}L(u)\zeta^2(\alpha+1),$$

by Lemma 1 and the choice of m_0. (The above computation is valid if $\int_0^R N(u)u^{-1}du$ exists in the Lebesgue sense for every positive R. This is indeed the case under the assumptions of the theorem since

$$\int_0^R \frac{N(u)}{u} du \leqq \int_0^R \frac{1}{u}\left(n(u) + \int_0^u \frac{n(t)}{t} dt\right) du$$

$$= \int_0^R \frac{n(u)}{u} du + \int_0^R \left(\int_t^R \frac{du}{u}\right)\frac{n(t)}{t} dt$$

$$= \int_0^R \frac{n(u)}{u} du + \int_0^R \frac{n(t)}{t} \log \frac{R}{t} dt,$$

which exist by assumption.) Combining the preceding results we obtain

$$\left| n(u) - u^{\alpha}L(u) \right| = \left|\sum_{k=1}^{\infty} \frac{\mu(k)}{k} N\left(\frac{u}{k}\right) - u^{\alpha}L(u)\right|$$

$$< 4\epsilon\zeta^2(\alpha+1)u^{\alpha}L(u).$$

Since ϵ is arbitrary, our theorem follows.

4. **Additional properties of slowly oscillating functions.** For the purposes of completing the following theorems it will be necessary to introduce a relationship between the variable u which tends to infinity and the variable s which tends to zero. That is we must define the variable s in terms of u. To this end we prove the following lemmas.

LEMMA 2. *Suppose A and α are positive constants and $J(u)$ is a normalized slowly oscillating function defined for $u \geqq u_0$,*

$$J(u) = K \exp \int_{u_0}^{u} \frac{\delta(t)}{t} dt,$$

where K is a positive constant and $\delta(u)$ is a continuous function which tends to zero as $u \to \infty$. Then for every sufficiently large u there is a unique positive number σ_u such that

$$u = A\left(\frac{1}{\sigma_u}\right)^{\alpha+1} J\left(\frac{1}{\sigma_u}\right).$$

In addition we have the following:

(a) *σ_u tends monotonically to zero as $u \to \infty$,*
(b) *σ_u is a differentiable function of u and*

$$\frac{d\sigma_u}{du} \sim \frac{-\sigma_u}{u(\alpha+1)}, \qquad \frac{d}{du}(u\sigma_u) \sim \sigma_u\left(\frac{\alpha}{\alpha+1}\right) \text{ as } u \to \infty,$$

(c) $u^{-1/(\alpha+1)}\left\{AJ(1/\sigma_u)\right\}^{1/(\alpha+1)} = \sigma_u = u^{-1/(\alpha+1)}L_1(u)$ *where $L_1(u)$ is a slowly oscillating function,*
(d) *If σ_u is defined for $u \geqq u_0$, then*

$$\int_{u_0}^{u} \sigma_t dt \sim \left(1 + \frac{1}{\alpha}\right) u\sigma_u \text{ as } u \to \infty.$$

Proof. Since the expression $A(1/\sigma)^{\alpha+1}J(1/\sigma)$ is a continuous function of σ which tends to ∞ as $\sigma \to 0$ (by property (iii) of a slowly oscillating function) and since

$$\frac{d}{d\sigma}\left\{A\left(\frac{1}{\sigma}\right)^{\alpha+1} J\left(\frac{1}{\sigma}\right)\right\} = A\left(\frac{1}{\sigma}\right)^{\alpha+1} J\left(\frac{1}{\sigma}\right)\left\{\frac{-(\alpha+1)}{\sigma} - \frac{\delta(1/\sigma)}{\sigma}\right\} < 0$$

for small σ, the existence and uniqueness of σ_u for large u, as well as the properties (a) and (b) follow immediately from the inverse function theorem.

To prove (c) we note that by (b) we have for arbitrary positive ϵ and large u

$$\frac{-(1+\epsilon)\sigma_u}{u(\alpha+1)} \leqq \frac{d\sigma_u}{du} \leqq \frac{-(1-\epsilon)\sigma_u}{u(\alpha+1)}.$$

Hence for fixed $H \geqq 1$ and large u, we have

$$\frac{-(1+\epsilon)}{\alpha+1}\int_u^{Hu} \frac{du}{u} \leqq \int_u^{Hu} \frac{1}{\sigma_u}\frac{d\sigma_u}{du} \leqq \frac{-(1-\epsilon)}{\alpha+1}\int_u^{Hu}\frac{du}{u}$$

or

$$(1-\epsilon)\log H^{1/(\alpha+1)} \leqq \log\frac{\sigma_u}{\sigma_{Hu}} \leqq (1+\epsilon)\log H^{1/(\alpha+1)}.$$

If $H<1$ we reverse the limits on the integrals, obtaining the same result with the inequality signs reversed. Since ϵ is arbitrary, $\lim_{u\to\infty}\log(\sigma_u/\sigma_{Hu}) = \log H^{1/(\alpha+1)}$ or $\sigma_{Hu}/\sigma_u \sim H^{-1/(\alpha+1)}$ for large u. We now let $L_1(u) = u^{1/(\alpha+1)}\sigma_u$. It is then clear that $L_1(u)$ is a slowly oscillating function.

In order to establish (d) we write

$$\int_{u_0}^{u}\sigma_t dt = \int_{u_0}^{u} t^{(\alpha/(\alpha+1))-1}L_1(t)dt \sim \frac{(\alpha+1)}{\alpha}u^{\alpha/(\alpha+1)}L_1(u) = \left(1+\frac{1}{\alpha}\right)u\sigma_u$$

as $u \to \infty$,

where we have made use of property (c) to establish the first and third relations and of Lemma 1 to establish the second.

In the case of a more general slowly oscillating function than the one considered above we are able to assert only that s_u is asymptotic and not necessarily equal to $u^{-1/(\alpha+1)}L_1(u)$, where $L_1(u)$ is a slowly oscillating function. This is because $u^{1/(\alpha+1)}s_u$ may not be continuous in general, since $A(1/s)^{\alpha+1}L(1/s)$ may not be monotonic in the small.

LEMMA 3. *Suppose A and α are positive constants and $L(u)$ is any slowly oscillating function. Then for every sufficiently large u, there exists a positive number s_u such that*

$$u = A\left(\frac{1}{s_u}\right)^{\alpha+1} L\left(\frac{1}{s_u}\right).$$

In addition we have the following:

(a) *s_u tends to zero as $u \to \infty$,*

(b) *s_u is determined up to a factor which tends to one as $u \to \infty$,*

(c) *There is a slowly oscillating function $L_1(u)$ such that*

$$s_u = u^{-1/(\alpha+1)}\left\{AL\left(\frac{1}{s_u}\right)\right\}^{1/(\alpha+1)} \sim u^{-1/(\alpha+1)}L_1(u) \text{ as } u \to \infty .$$

In fact, if $L(u) = (1+r(u))J(u)$ where $J(u)$ is a normalized slowly oscillating function and if σ_u is as defined in Lemma 2 relative to J, then $u^{-1/(\alpha+1)}L_1(u) = \sigma_u$.

Proof. Since the expression $A(1/s)^{\alpha+1}L(1/s)$ is a continuous function of s which tends to infinity as s tends to zero, the existence of s_u for large u and the property (a) are immediate. By property (i) of a slowly oscillating function, $L(u)$ must have the form $(1+r(u))J(u)$ where $J(u)$ is of the normalized form discussed in the preceding lemma. Hence

$$u = A\left(\frac{1}{s_u}\right)^{\alpha+1}\left(1 + r\left(\frac{1}{s_u}\right)\right)J\left(\frac{1}{s_u}\right).$$

Thus for large u

$$s_u = \sigma_{u/(1+r(1/s_u))} = \left(\frac{u}{1 + r(1/s_u)}\right)^{-1/(\alpha+1)} L_1\left(\frac{u}{1 + r(1/s_u)}\right) \sim u^{-1(\alpha+1)}\, L_1(u)$$
$$\text{as } u \to \infty$$

where $L_1(u)$ is as obtained in the previous lemma. This shows the validity of (b) and (c).

5. The connection between $\log f(s)$ **and** $\log P(u)$:

THEOREM 3. *Suppose $\int_0^R P(u)du$ exists in the Lebesgue sense for every posi-*

tive R, $f(s)=s\int_0^\infty P(u)e^{-su}du$, for all positive s, B and α are positive constants, L is a slowly oscillating function, and s_u is a positive number defined for large u such that $u=B\alpha(1/s_u)^{\alpha+1}L(1/s_u)$. Then if $P(u)$ is nondecreasing, the relation $\log f(s)\sim B(1/s)^\alpha L(1/s)$ *as $s\to 0$ implies the relation* $\log P(u)\sim(1+1/\alpha)us_u$ *as $u\to\infty$.*

Proof. Since adding a positive constant to $P(u)$ merely adds a constant to $f(s)$, we may assume $P(u)\geqq 0$ for all positive u. Property (i) of a slowly oscillating function and Lemma 3 (c) shows that there is no loss of generality in assuming $L(1/s)$ of the special form

$$J\left(\frac{1}{s}\right) = K \exp \int_1^{1/s} \frac{\delta(t)}{t}\,dt.$$

We shall make this assumption throughout this proof. Let ϵ be an arbitrary positive number less than 1. For any $u>0$, $s>0$ we have

$$P(u)e^{-su} = s\int_u^\infty P(u)e^{-sx}dx \leqq s\int_u^\infty P(x)e^{-sx}dx$$

since $u\leqq x$ and $P(u)$ is nondecreasing. Thus, since $P(u)$ is non-negative

$$P(u)e^{-su} \leqq s\int_0^\infty P(x)e^{-sx}dx = f(s).$$

Hence, by the hypothesis on $f(s)$, we have for all sufficiently small s

$$P(u) < \exp\left\{(1+\epsilon)B\left(\frac{1}{s}\right)^\alpha L\left(\frac{1}{s}\right) + su\right\}.$$

The estimate will be about the best possible if we choose s so that $(1/s)^\alpha L(1/s)$ $+su$ is about as small as possible. Since $\delta(1/s)\to 0$ as $s\to 0$ and

$$\frac{d}{ds}\left\{B\left(\frac{1}{s}\right)^\alpha L\left(\frac{1}{s}\right) + su\right\} = -B\alpha\left(\frac{1}{s}\right)^{\alpha+1} L\left(\frac{1}{s}\right)\left(1+\frac{\delta(1/s)}{\alpha}\right) + u,$$

this minimum occurs, for large u, when s is near that value such that u $=B\alpha(1/s)^{\alpha+1}L(1/s)$. This is the value of s denoted by s_u in the statement of the theorem. Using this value of s, and replacing $B(1/s_u)^\alpha L(1/s_u)$ by us_u/α we obtain

$$P(u) < \exp\left\{(1+\epsilon)\left(1+\frac{1}{\alpha}\right)us_u\right\}$$

for large u.

To get a lower estimate for $P(u)$ for large u we also consider $f(s_u)$. We split $f(s_u)=s_u\int_0^\infty P(x)e^{-xs_u}dx$ into five parts

$$s_u\int_0^{u/H}+s_u\int_{u/H}^{(1-\zeta)u}+s_u\int_{(1-\zeta)u}^{(1+\zeta)u}+s_u\int_{(1+\zeta)u}^{Hu}+s_u\int_{Hu}^{\infty}$$

where H is a fixed positive number such that

$$H \geqq \max \left\{1, (8\alpha+8)^{1+1/\alpha}, (8\alpha^{-1}+8)^{\alpha+1}\right\}$$

and ζ is a positive number less than 1, an exact choice of which (in terms of ϵ) will be made later ($\zeta=\epsilon/3$).

The five parts of $f(s_u)$ we denote by J_1, J_2, J_3, J_4, J_5, respectively. We shall use the upper estimate for $P(u)$ obtained above to show that the contributions to J_1, J_2, J_4, and J_5 to $f(s_u)$ are comparatively small.

Now $P(x)<\exp\left\{2(1+1/\alpha)xs_x\right\}$ for large x and xs_x is increasing for large x (by Lemma 2(b)) and tends to infinity with x. Hence we have for large u

$$J_1 = s_u\int_0^{u/H}P(x)e^{-xs_u}dx \leqq \exp\left\{2\left(1+\frac{1}{\alpha}\right)\frac{u}{H}s_{u/H}\right\}s_u\int_0^{\infty}e^{-xs_u}dx$$
$$\leqq \exp\left\{4\left(1+\frac{1}{\alpha}\right)H^{-\alpha/(\alpha+1)}us_u\right\}$$

since $s_{u/H}\sim H^{1/(\alpha+1)}s_u$ by Lemma 2(c). Hence $J_1\leqq\exp\left\{us_u/2\alpha\right\}$, since H has been chosen so that $H\geqq(8(\alpha+1))^{1+1/\alpha}$.

For $x\geqq Hu$, where u is sufficiently large, we have $P(x)<\exp\left\{2(1+1/\alpha)xs_u\right\}$. We claim $P(x)\leqq\exp\left\{xs_u/2\right\}$ for $x\geqq Hu$. It suffices to show that $2(1+1/\alpha)xs_x \leqq xs_u/2$ for $x\geqq Hu$. Since s_x is a decreasing function of x for large x, and hence $s_{Hu}\geqq s_x$, for $x\geqq Hu$, it suffices to show $4(1+1/\alpha)s_{Hu}\leqq s_u$. Since by Lemma 2(c), $s_{Hu}\sim H^{-1/(\alpha+1)}s_u$ and thus $s_{Hu}/s_u\leqq 2H^{-1/(\alpha+1)}$ for large u, we need only show $2H^{-1/(\alpha+1)}\leqq 1/4(1+1/\alpha)$. This clearly holds since H has been chosen $\geqq(8(1+1/\alpha))^{\alpha+1}$. Thus we have

$$J_5 = s_u\int_{Hu}^{\infty}P(x)e^{-xs_u}dx \leqq s_u\int_{Hu}^{\infty}e^{xs_u/2}e^{-xs_u}dx \leqq s_u\int_0^{\infty}e^{-xs_u/2}dx = 2.$$

We now define $\vartheta(x)$ so that $\vartheta'(x)=s_x$, that is we take $\vartheta(x)=\int_{t_0}^{x}s_t dt$, where t_0 is such that s_t is defined for all $t\geqq t_0$. Now by Lemma 2, $\vartheta(x)\sim(1+1/\alpha)xs_x$ and $\vartheta''(x)\sim -s_x/x(\alpha+1)$. We assume u so large that $\vartheta(x)\leqq(1+2\eta)/(1+\eta)\cdot(1+1/\alpha)xs_x$ and $P(x)\leqq\exp\left\{(1+\eta)\vartheta(x)\right\}$ for $x\geqq u/H$, where η is a positive number less than 1, to be chosen later in terms of ζ, H and α. The integrand in J_2 and J_4 does not exceed $\exp\left\{(1+\eta)\vartheta(x)-xs_u\right\}$ on the interval $[u/H, Hu]$. Let $\phi(x)=(1+\eta)\vartheta(x)-xs_u$. Applying Taylor's Theorem to $\phi(x)$ around $x=u$ (the maximum value of $\phi(x)$ occurs near $x=u$), we have

$$\phi(x) = \phi(u) + (x-u)\phi'(u) + \frac{(x-u)^2}{2}\phi''(u+\theta(x-u)) \text{ where } 0<\theta<1.$$

Now

$$\phi(u) = (1+\eta)\vartheta(u) - us_u \leqq (1+2\eta)\left(1+\frac{1}{\alpha}\right)us_u - us_u$$
$$= \frac{us_u}{\alpha} + 2\eta\left(1+\frac{1}{\alpha}\right)us_u$$

and $\phi'(u)=(1+\eta)\vartheta'(u)-s_u=\eta s_u$.

Assume u so large that $\vartheta''(x)\leqq(-1/(1+\eta))(s_x/x(\alpha+1))$ for $x\geqq u/H$. Then if $u/H\leqq x\leqq Hu$ we have

$$\phi''(x) = (1+\eta)\vartheta''(x) \leqq \frac{-s_x}{x(\alpha+1)} \leqq \frac{-s_{Hu}}{x(\alpha+1)} \leqq \frac{-s_{Hu}}{Hu(\alpha+1)}.$$

Now $s_{Hu}\sim H^{-1/\alpha+1}s_u$ as $u\to\infty$ and so for sufficiently large u, we have

$$s_{Hu} \geqq H^{-1}s_u.$$

Then if $u/H\leqq x\leqq Hu$ we have

$$\phi''(x) \leqq \frac{-s_u}{H^2u(\alpha+1)}.$$

Thus if x is in $[(u/H), Hu]$ we have

$$\phi(x) \leqq \frac{us_u}{\alpha} + 2\eta\left(1+\frac{1}{\alpha}\right)us_u + H\eta us_u - \frac{s_u}{H^2u(\alpha+1)}\,\frac{(x-u)^2}{2}.$$

If in addition $|x-u|\geqq\zeta u$ then

$$\phi(x) \leqq \frac{us_u}{\alpha} + 2\eta\left(1+\frac{1}{\alpha}\right)us_u + H\eta us_u - \frac{\zeta^2 us_u}{2H^2(\alpha+1)}$$
$$\leqq \frac{us_u}{\alpha} + \left\{\left(2+\frac{2}{\alpha}+H\right)\eta - \frac{\zeta^2}{2H^2(\alpha+1)}\right\}us_u$$
$$\leqq \frac{us_u}{\alpha} - \frac{\zeta^2}{3H^2(\alpha+1)}\,us_u$$

provided we take

$$\eta = \frac{\zeta^2}{6H^2(\alpha+1)(2+(2/\alpha)+H)}.$$

Hence if u is sufficiently large, we have

$$J_2 + J_4 \leqq s_u\int_{u/H}^{(1-\zeta)u} e^{\phi(x)}dx + s_u\int_{(1+\zeta)u}^{Hu} e^{\phi(x)}dx$$
$$\leqq Hus_u \exp\left\{\left(\frac{1}{\alpha} - \frac{\zeta^2}{3H^2(\alpha+1)}\right)us_u\right\}.$$

Combining our estimates, we obtain for u sufficiently large

$$J_1 + J_2 + J_4 + J_5 \leqq 3 \exp \left\{ \frac{us_u}{2\alpha} \right\} + Hus_u \exp \left\{ \left(\frac{1}{\alpha} - \frac{\zeta^2}{3H^2(\alpha+1)} \right) us_u \right\}$$

$$\leqq (H+3)us_u \exp \left\{ \left(\frac{1}{\alpha} - \frac{\zeta^2}{3H^2(\alpha+1)} \right) us_u \right\}$$

$$< \exp \left\{ \left(\frac{1}{\alpha} - \frac{\zeta^2}{4H^2(\alpha+1)} \right) us_u \right\}$$

$$< \exp \left\{ \left(\frac{1}{\alpha} - 2\delta \right) us_u \right\}, \qquad \text{where } \delta = \frac{\zeta^2}{8H^2(\alpha+1)}.$$

Now by hypothesis we have for s_u sufficiently small, i.e., for u sufficiently large

$$f(s_u) \geqq \exp \left\{ (1 - \alpha\delta) B \left(\frac{1}{s_u} \right)^{\alpha} L \left(\frac{1}{s_u} \right) \right\}$$

$$\geqq \exp \left\{ (1 - \alpha\delta) \frac{us_u}{\alpha} \right\} = \exp \left\{ \left(\frac{1}{\alpha} - \delta \right) us_u \right\}.$$

Thus

$$J_3 = f(s_u) - (J_1 + J_2 + J_3 + J_4 + J_5) \geqq \exp \left\{ \left(\frac{1}{\alpha} - \delta \right) us_u \right\}$$

$$- \exp \left\{ \left(\frac{1}{\alpha} - 2\delta \right) us_u \right\} \geqq \exp \left\{ \left(\frac{1}{\alpha} - 2\delta \right) us_u \right\}$$

for u sufficiently large. Now by the monotonicity of $P(x)$ we have

$$J_3 = s_u \int_{(1-\zeta)u}^{(1+\zeta)u} P(x) e^{-xs_u} dx \leqq P((1+\zeta)u) s_u \int_{(1-\zeta)u}^{(1+\zeta)u} e^{-xs_u} dx$$

$$< P((1+\zeta)u) \exp \left\{ -(1-\zeta)us_u \right\}.$$

Hence for every sufficiently large u, we have

$$P((1+\zeta)u) \geqq J_3 \exp \left\{ (1-\zeta)us_u \right\} \geqq \exp \left\{ \left(\frac{1}{\alpha} - 2\delta + 1 - \zeta \right) us_u \right\}$$

$$\geqq \exp \left\{ \left(1 + \frac{1}{\alpha} \right) (1 - 2\delta - \zeta) us_u \right\} \geqq \exp \left\{ \left(1 + \frac{1}{\alpha} \right) (1 - 2\zeta) us_u \right\}.$$

Thus $P((1+\zeta)u) \geqq \exp \left\{ (1+1/\alpha)(1-3\zeta)(1+\zeta)us_u \right\}$ for u large, since $s_u \geqq s_{(1+\zeta)u}$. On replacing $(1+\zeta)u$ by y we have, for y sufficiently large, $P(y) \geqq \exp \left\{ (1-3\zeta)(1+1/\alpha)ys_y \right\}$.

Now we take $\zeta=\epsilon/3$. Then for large y,

$$P(y) > \exp \{(1 - \epsilon)(1 + 1/\alpha)ys_y\},$$

and this completes the proof.

THEOREM 4. *Suppose* $\int_0^R P(u)du$ *exists in the Lebesgue sense for every positive* R, $f(s)=s\int_0^\infty P(u)e^{-su}du$, *for all positive* s, B *and* α *are positive constants,* L *is a slowly oscillating function, and* s_u *is a positive number defined for large* u *such that* $u=B\alpha(1/s_u)^{\alpha+1}L(1/s_u)$. *Then the relation* $\log P(u)\sim(1+1/\alpha)us_u$ *as* $u\to\infty$ *implies the relation* $\log f(s)\sim B(1/s)^\alpha L(1/s)$ *as* $s\to 0$.

Proof. As in the proof of Theorem 3, we may assume that L is a normalized slowly oscillating function, so that Lemma 2 can be applied. We may also assume that $P(x)\geqq 0$ for all positive x. Let $0<\epsilon<1/(2\alpha+3)$ be given. Then $P(x)>\exp\{(1-\epsilon)(1+1/\alpha)xs_x\}$ for sufficiently large x. Hence if u is sufficiently large

$$\begin{aligned} f(s_u) &\geqq s_u\int_u^{(1+\epsilon)u}\exp\left\{(1-\epsilon)\left(1+\frac{1}{\alpha}\right)xs_x - xs_u\right\}dx \\ &\geqq \exp\left\{(1-\epsilon)\left(1+\frac{1}{\alpha}\right)us_u\right\}s_u\int_u^{(1+\epsilon)u}\exp\{-xs_u\}dx \\ &= \exp\left\{(1-\epsilon)\left(1+\frac{1}{\alpha}\right)us_u\right\}\exp\{-us_u\}(1-\exp\{-\epsilon us_u\}) \\ &\geqq \exp\left\{(1-2\epsilon)\left(1+\frac{1}{\alpha}\right)us_u - us_u\right\}, \end{aligned}$$

since $\exp\{-\epsilon us_u\}\to 0$ as $u\to\infty$. Thus for u sufficiently large

$$\log f(s_u) \geqq \{1-(2\alpha+3)\epsilon\}\frac{us_u}{\alpha} = \{1-(2\alpha+3)\epsilon\}B\left(\frac{1}{s_u}\right)^\alpha L\left(\frac{1}{s_u}\right).$$

We proceed now to obtain the estimate from above. We split $f(s_u)$ into the same five parts as in the proof of Theorem 3, and use here the estimates from above for J_1, J_2, J_4, and J_5, where H and ζ are chosen as before. This is permissible since nowhere in the computation of these estimates did we make use of the monotonicity of $P(u)$. Indeed we used only the fact that $P(u) <\exp\{(1+\epsilon)(1+1/\alpha)us_u\}$ for sufficiently large n, which is now immediate by hypothesis. Here

$$J_3 = s_u\int_{(1-\zeta)u}^{(1+\zeta)u}P(x)e^{-xs_u}dx \leqq s_u\int_{(1-\zeta)u}^{(1+\zeta)u}\exp\{(1+\epsilon)(1+1/\alpha)xs_x - xs_u\}dx$$

for u sufficiently large. Hence

$$J_3 \leqq \exp\left\{(1+\epsilon)\left(1+\frac{1}{\alpha}\right)(1+\zeta)us_{(1+\zeta)u}\right\} s_u \int_{(1-\zeta)u}^{(1+\zeta)u} \exp\{-xs_u\}\,dx$$

$$\leqq \exp\left\{(1+\epsilon)\left(1+\frac{1}{\alpha}\right)(1+\zeta)us_u\right\} \exp\{-(1-\zeta)us_u\}$$

since $s_{(1+\zeta)u} \leqq s_u$ for u sufficiently large. With $\zeta=\epsilon/3$ this gives

$$J_3 \leqq \exp\{(1/\alpha)(1+(2\alpha+2)\epsilon)us_u\}.$$

Now, using this inequality and the estimates for J_1, J_2, J_4, J_5, obtained in the proof of Theorem 3, we have for sufficiently large u,

$$f(s_u) \leqq \exp\left\{\left(\frac{1}{\alpha}-2\delta\right)us_u\right\} + \exp\left\{(1+(2\alpha+2)\epsilon)\frac{us_u}{\alpha}\right\}$$

$$\leqq 2\exp\left\{(1+(2\alpha+2)\epsilon)\frac{us_u}{\alpha}\right\}$$

$$\leqq \exp\left\{(1+(2\alpha+3)\epsilon)\frac{us_u}{\alpha}\right\},$$

$$\log f(s_u) \leqq \{1+(2\alpha+3)\epsilon\}\frac{us_u}{\alpha} = \{1+(2\alpha+3)\epsilon\} B\left(\frac{1}{s_u}\right)^{\alpha} L\left(\frac{1}{s_u}\right).$$

This inequality, together with the estimation of $\log f(s_u)$ from below gives the desired result, since when u runs through all sufficiently large positive values, s_u takes all sufficiently small positive values by Lemma 2.

6. **Statement of results.** The results obtained may be summarized in the following manner.

MAIN THEOREM. *Suppose that $n(u)$ and $P(u)$ are functions on the non-negative reals and that $\int_0^R n(u)u^{-1}du$ and $\int_0^R n(u)u^{-1}\log u\,du$ and $\int_0^R P(u)du$ exist in the Lebesgue sense for every positive R. Suppose further that*

$$\exp\left\{s\int_0^\infty \frac{e^{-su}}{1-e^{-su}}\,n(u)du\right\} = s\int_0^\infty P(u)e^{-su}du$$

for all positive s. Suppose that α is a positive constant and that $L(u)$ is a slowly oscillating function. For large u let s_u be a positive number such that

$$u = \alpha\Gamma(\alpha+1)\zeta(\alpha+1)\left(\frac{1}{s_u}\right)^{\alpha+1} L\left(\frac{1}{s_u}\right)$$

and let $L^*(u)$ be a slowly oscillating function defined for large u such that $L^*(u)\sim(1+1/\alpha)\{\alpha\Gamma(\alpha+1)\zeta(\alpha+1)L(1/s_u)\}^{1/(\alpha+1)}$, the existence of $L^*(u)$ being guaranteed by Lemma 3(c).

(i) *If $n(u)\sim u^\alpha L(u)$ as $u\to\infty$ and $P(u)$ is nondecreasing, then* $\log P(u)$ $\sim(1+1/\alpha)us_u\sim u^{\alpha/(\alpha+1)}L^*(u)$ *as $u\to\infty$.*

(ii) *If* $\log P(u) \sim (1+1/\alpha)us_u \sim u^{\alpha/(\alpha+1)}L^*(u)$ *as* $u \to \infty$ *and if* $n(u)$ *is nondecreasing, then* $n(u) \sim u^\alpha L(u)$ *as* $u \to \infty$.

Proof. (i) follows Theorems 1 and 3, while (ii) is the result of Theorems 4 and 2.

In the case of partitions the functions $n(u)$ and $P(u)$ are automatically nondecreasing and trivially satisfy the required conditions of integrability. Thus we have the following corollary, as mentioned in the introduction.

COROLLARY 1. *Suppose* $n(u)$ *and* $P(u)$ *are defined from a set* Λ *of positive real numbers and a non-negative valued function* $\psi(k)$ *on the positive integers as in the second paragraph of the introduction. Suppose* α, s_u, $L(u)$, *and* $L^*(u)$ *are as in the Main Theorem. Then* $n(u) \sim u^\alpha L(u)$ *as* $u \to \infty$ *if and only if* $\log P(u) \sim (1+1/\alpha)us_u \sim u^{\alpha/\alpha+1}L^*(u)$ *as* $u \to \infty$.

It should be pointed out that in any special case of the function $L(u)$ it will usually be possible to "solve" the equation $u = \alpha\Gamma(\alpha+1)\zeta(\alpha+1) \cdot (1/s_u)^{\alpha+1}L(1/s_u)$, to obtain an asymptotic formula for s_u in terms of u. As an example, consider the function $L(u) = K(\log u)^{a_1}(\log\log u)^{a_2} \cdots (\log_k u)^{ak}$, $\log_k u$ denoting the "kth iterated logarithm of u." This function is slowly oscillating. (See [10, p. 68].) Since $\log u \sim (\alpha+1) \log(1/s_u)$ we have in this case $L(1/s_u) = (1/(\alpha+1))^{a_1}L(u)$. Thus if $L(u)$ has this special form, the conclusion of the above corollary reduces to:

COROLLARY 1*. *Suppose* $n(u)$ *and* $P(u)$ *are defined from a set* Λ *of positive real numbers and a non-negative valued function* $\psi(k)$ *on the positive integers as in the second paragraph of the introduction. Suppose* α *and* s_u *are as in the Main Theorem while* $L(u)$ *is of the form* $K(\log u)^{a_1}(\log\log u)^{a_2} \cdots (\log_k u)^{ak}$. *Then,* $n(u) \sim u^\alpha L(u)$ *as* $u \to \infty$ *if and only if*

$$\log P(u) \sim \left(1 + \frac{1}{\alpha}\right)\left\{\alpha\Gamma(\alpha+1)\zeta(\alpha+1)\left(\frac{1}{\alpha+1}\right)^{a_1}\right\}^{1/(\alpha+1)} u^{\alpha/(\alpha+1)}L(u)^{1/(\alpha+1)}$$

$$\text{as } u \to \infty.$$

This includes the Knopp-Erdös theorem mentioned in the introduction as well as Brigham's theorem and its converse as very special cases.

We remark that if $\lambda_1, \lambda_2, \cdots$, are positive *integers* and if the greatest common divisor of those λ_i for which $\psi(i) \geqq 1$ is unity, then an asymptotic relation for $\log P(u)$ of the sort given in Corollary 1 or Corollary 1* is equivalent to the same asymptotic relation for $\log p(n)$ as n goes to infinity through *integral* values. In fact, since the additive semigroup generated by a set of coprime positive integers contains all sufficiently large positive integers (see Knopp [7, pp. 60–63] and Ostmann [9, pp. 25–26]), it follows that there is a fixed positive integer c such that every integer not less than c is expressible in the form $m_1\lambda_{j_1} + m_2\lambda_{j_2} + \cdots$, where $m_1, m_2, \cdots$ are non-negative integers and $\psi(j_1) \geqq 1, \psi(j_2) \geqq 1, \cdots$. From this it is easy to see that $p(n) \geqq p(m)$ for any integers n and m with $n-c \geqq m \geqq 0$. Hence

$$P(n) \geqq p(n) \geqq \frac{P(n-c)}{n-c+1}$$

for integral $n \geqq c$, from which the asserted equivalence readily follows.

When $\psi(k)=1$ for all positive integral values of k, the generating function $f(s)=\prod_{k=1}^{\infty}(1-e^{-s\lambda_k})^{-\psi(k)}$ discussed in this paper relates to partitions into parts taken from a set $\Lambda=\{\lambda_1, \lambda_2, \cdots\}$, repetitions being allowed. One is led to ask whether or not similar conclusions might be obtained for partitions into *distinct* parts taken from Λ. The generating function in the latter case is $g(s)=\prod_{k=1}^{\infty}(1+e^{-s\lambda_k})=\sum_{m=0}^{\infty} q(\nu_m)e^{-s\nu_m}$, say, where $\nu_0, \nu_1, \cdots$ are again the elements of the additive semi-group generated by Λ.

More generally we have the following result. The special case in which $L(u)$ is a constant was proved by Knopp [7].

COROLLARY 2. *Suppose $\psi(k)$ is a function on the positive integers taking non-negative integral values and is such that $\sum_{\lambda_k \leqq u} \psi(k) \sim u^{\alpha}L(u)$ as $u \to \infty$, where α is a positive constant and $L(u)$ is a slowly oscillating function. Suppose $\prod_{k=1}^{\infty}(1+e^{-s\lambda_k})^{\psi(k)}=\sum_{m=0}^{\infty} q(\nu_m)e^{-s\nu_m}$ and $Q(u)=\sum_{\nu_m \leqq u} q(\nu_m)$. For large u let s_u be a positive number such that $u=(1-1/2^{\alpha})\alpha\Gamma(\alpha+1)\zeta(\alpha+1)(1/s_u)^{\alpha+1}L(1/s_u)$ and let $L^{**}(u)$ be a slowly oscillating function such that*

$$L^{**}(u) \sim \left(1+\frac{1}{\alpha}\right)\left\{\left(1-\frac{1}{2^{\alpha}}\right)\alpha\Gamma(\alpha+1)\zeta(\alpha+1)L\left(\frac{1}{s_u}\right)\right\}^{1/(\alpha+1)} \text{ as } u \to \infty.$$

Then

$$\log Q(u) \sim (1+1/\alpha)us_u \sim u^{\alpha/(\alpha+1)}L^{**}(u) \qquad \text{as } u \to \infty.$$

Proof. If $g(s)$ is the generating function here and $f(s)=\prod_{k=1}^{\infty}(1-e^{-s\lambda_k})^{-\psi(k)}$ then $g(s)=f(s)/f(2s)$. By Theorem 1 we can conclude that $\log g(s) \sim(1-1/2^{\alpha})\Gamma(\alpha+1)\zeta(\alpha+1)(1/s)^{\alpha}L(1/s)$ as $s \to 0$. Under the assumptions on ψ we note that $Q(u)$ is nondecreasing and so, since $g(s)=s\int_0^{\infty} Q(u)e^{-su}du$, the result follows from Theorem 3.

COROLLARY 2*. *Suppose ψ, q, Q and s_u are as in Corollary 2 and $L(u)$ is of the form $K(\log u)^{a_1}(\log\log u)^{a_2}\cdots(\log_k u)^{a_k}$. Then if $\sum_{\lambda_k \leqq u}\psi(k) \sim u^{\alpha}L(u)$ as $u \to \infty$, we have*

$$\log Q(u) \sim \left(1+\frac{1}{\alpha}\right) \cdot \left\{\left(1-\frac{1}{2^{\alpha}}\right)\alpha\Gamma(\alpha+1)\zeta(\alpha+1)\left(\frac{1}{\alpha+1}\right)^{a_1}\right\}^{1/(\alpha+1)} u^{\alpha/(\alpha+1)}L(u)^{1/(\alpha+1)} \text{ as } u \to \infty.$$

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